EFFICIENT COMPUTATION OF RESONANCE VARIETIES VIA GRASSMANNIANS

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Abstract. Associated to the cohomology ring A of the complement $X(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ are the resonance varieties $R^k(A)$. The most studied of these is $R^1(A)$, which is the union of the tangent cones at $\mathbf{1}$ to the characteristic varieties of $\pi_1(X(\mathcal{A}))$. $R^1(A)$ may be described in terms of Fitting ideals, or as the locus where a certain Ext module is supported. Both these descriptions give obvious algorithms for computation. In this note, we show that interpreting $R^1(A)$ as the locus of decomposable two-tensors in the Orlik-Solomon ideal of $\mathcal{A}$ leads to a description of $R^1(\mathcal{A})$ as the intersection of a Grassmannian with a linear space, determined by the quadratic generators of the Orlik-Solomon ideal. This method is much faster than previous alternatives.

1. Motivation: cohomology rings of arrangement complements

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central complex hyperplane arrangement in $\mathbb{C}^\ell$, and let $X(\mathcal{A}) = \mathbb{C}^\ell \setminus \mathcal{A}$. In [16], Orlik and Solomon determined a presentation for $A = H^*(X(\mathcal{A}), \mathbb{Z})$:

Definition 1.1. *$A = H^*(X(\mathcal{A}), \mathbb{Z})$ is the quotient of the exterior algebra $E = \bigwedge(\mathbb{Z}^n)$ on generators $e_1, \ldots, e_n$ in degree 1 by the ideal I generated by all elements of the form $\partial e_{i_1 \ldots i_r} := \sum_q (-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r}$, for which* $\operatorname{codim} H_{i_1} \cap \cdots \cap H_{i_r} < r$.

Since A is a quotient of an exterior algebra, multiplication by an element $a \in A^1$ gives a degree one differential on A, yielding a cochain complex (A, a):

$$(A, a): \quad 0 \longrightarrow A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots \xrightarrow{a} A^\ell \longrightarrow 0 . \tag{1.1}$$

Aomoto [1] studied this complex in connection with his work on hypergeometric functions, and the complex was subsequently studied in relation to local system cohomology by Esnault, Schechtman and Viehweg in [8]. The complex (A, a) is exact as long as $\sum_{i=1}^n a_i \neq 0$, and in [22], Yuzvinsky showed that in fact (A, a) is generically exact except at the last position ℓ.

Fix a field $\Bbbk$, we will write $A = H^*(X(\mathcal{A}), \Bbbk)$ for the Orlik-Solomon algebra over $\Bbbk$. The *resonance varieties* of $\mathcal{A}$ consist of points $a = \sum_{i=1}^n a_i e_i \leftrightarrow (a_1 : \cdots : a_n)$ in $\mathbb{P}(A^1) \cong \mathbb{P}^{n-1}$ for which (A, a) fails to be exact. So for each $k \geq 1$,

$$R^k(\mathcal{A}) = \{a \in \mathbb{P}^{n-1} \mid H^k(A, a) \neq 0\}.$$

Falk initiated the study of $R^1(\mathcal{A})$ in [9], obtaining necessary and sufficient combinatorial conditions for $a \in R^1(A)$. Falk also conjectured that $R^1(\mathcal{A})$ is the union of

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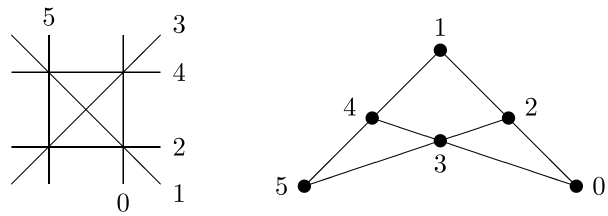

FIGURE 1. The braid arrangement and its matroid

a subspace arrangement. This was proved in [4]. By using the Cartan classification of affine Kac-Moody Lie algebras, Libgober and Yuzvinsky [14] also obtained this result, and showed that $R^1(\mathcal{A})$ is in fact a union of *disjoint*, positive dimensional subspaces. For the higher resonance varieties, Cohen and Orlik show in [2] that all components of the $R^k(\mathcal{A})$ are linear subvarieties. For all this, characteristic zero is necessary, see [10]. A major impetus for studying $R^1(A)$ is a conjecture of Suciu in [21], relating $R^1(A)$ to the LCS ranks of the fundamental group. Results on this conjecture appear in [13], [15], [18].

In [7], Eisenbud, Popescu, and Yuzvinsky prove that the complex (A, d_a), regarded as a complex of $S = Sym(\Bbbk^n)$ modules, is a free resolution of the cokernel $F(A)$ of the final nonzero map. Combining with results of [3],[4], the paper [20] shows that:

$$R^1(\mathcal{A}) = V(\mathrm{ann}(\mathrm{Ext}^{\ell-1}(F(A), S))).$$

In [5], this result is generalized to:

$$R^k(\mathcal{A}) = \bigcup_{k' \le k} V(\mathrm{ann}\,\mathrm{Ext}^{\ell-k'}(F(A), S))$$

In particular, the resonance varieties of hyperplane arrangements may be realized as support loci of appropriate Ext modules.

Example 1.2. Let $\mathcal{A}$ be the braid arrangement in $\mathbb{P}^2$, with defining polynomial $Q = xyz(x-y)(x-z)(y-z)$. From the matroid (see Figure 1), it is easy to see that the Orlik-Solomon algebra A is the quotient of the exterior algebra E on generators $e_0, \ldots, e_5$ by the ideal $I = \langle \partial e_{145}, \partial e_{235}, \partial e_{034}, \partial e_{012}, \partial e_{ijkl} \rangle$, where $ijkl$ runs over all four-tuples; it turns out that the elements ∂e_{ijkl} are redundant.

The minimal free resolution of A as a module over E begins:

$$0 \longleftarrow A \longleftarrow E \xleftarrow{\partial_1} E^4(-2) \xleftarrow{\partial_2} E^{10}(-3) \xleftarrow{\partial_3} E^{15}(-4) \oplus E^6(-5) \longleftarrow \cdots,$$

where $\partial_1 = \begin{pmatrix} \partial e_{145} & \partial e_{235} & \partial e_{034} & \partial e_{012} \end{pmatrix}$, and ∂_2 is equal to

$$\begin{pmatrix} e_1 - e_4 & e_1 - e_5 & 0 & 0 & 0 & 0 & 0 & 0 & e_3 - e_0 & e_2 - e_0 \\ 0 & 0 & e_2 - e_3 & e_2 - e_5 & 0 & 0 & 0 & 0 & e_0 - e_1 & e_0 - e_4 \\ 0 & 0 & 0 & 0 & e_0 - e_3 & e_0 - e_4 & 0 & 0 & e_1 - e_5 & e_2 - e_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_0 - e_1 & e_0 - e_2 & e_3 - e_5 & e_4 - e_5 \end{pmatrix}.$$

The resonance variety $R^1(\mathcal{A}) \subset \mathbb{P}^5$ has 4 local components, corresponding to the triple points, and 1 essential component (i.e., one that does not come from any

proper sub-arrangement), corresponding to the neighborly partition $\Pi = (05|13|24)$:

$$\{x_1 + x_4 + x_5 = x_0 = x_2 = x_3 = 0\},\ \{x_2 + x_3 + x_5 = x_0 = x_1 = x_4 = 0\},$$
$$\{x_0 + x_3 + x_4 = x_1 = x_2 = x_4 = 0\},\ \{x_0 + x_1 + x_2 = x_3 = x_4 = x_5 = 0\},$$
$$\{x_0 + x_1 + x_2 = x_0 - x_5 = x_1 - x_3 = x_2 - x_4 = 0\}.$$

The last two columns of the matrix representing ∂_2 correspond to a pair of linear syzygies on I_2, which arise from the essential component of $R^1(\mathcal{A})$:

$$\partial e_{012} + \partial e_{034} + \partial e_{145} - \partial e_{235} = (e_0 - e_1 - e_3 + e_5) \wedge (e_1 - e_2 + e_3 - e_4).$$

If we write the two-form above as $\lambda \wedge \mu = \sum a_i f_i \in I_2$, then these syzygies are:

$$0 = \lambda \wedge \lambda \wedge \mu = \sum a_i \lambda f_i \text{ and } 0 = \lambda \wedge \mu \wedge \mu = \sum a_i \mu f_i.$$

This example motivated investigations in [19] on the connection between $R^1(\mathcal{A})$ and the linear syzygies of A, where A is viewed as a module over the exterior algebra E. In this example, the syzygies arising from $R^1(\mathcal{A})$ are independent, but this is not the case in general.

This concludes our brief introduction to hyperplane arrangements and resonance varieties. For additional details on arrangements, see Orlik-Terao [17].

2. Grassmannians

We write $G(k, V)$ for the Grassmannian of k-planes in a vector space V. This is an affine cone, and can be thought of as the projective variety $\mathbb{G}(k-1, \mathbb{P}(V)))$. Let $\mathcal{W}_k \subset \mathbb{P}(E_1) \times \mathbb{P}(\Lambda^k E_1)$ denote the open subset $\mathcal{W}_k := \{([a], [\rho]) \mid a \wedge \rho \neq 0\}$. The various maps we need are displayed in the following diagram:

(2.1)
$$\begin{tikzcd} & \mathcal{W}_k \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "\mu_k"] & & \mathbb{P}(\Lambda^{k+1} E_1) \\ \mathbb{P}(E_1) & & \mathbb{P}(\Lambda^k E_1), & \end{tikzcd}$$

where μ_k denote the multiplication map $([a], [\rho]) \mapsto [a \wedge \rho]$ and the π_i's denote the projections.

Let $\Theta_k \subset \mathbb{P}(E_1) \times \mathbb{P}(\Lambda^k E_1) \times \mathbb{P}(\Lambda^{k+1} E_1)$ denote the graph of μ_k. If $\pi_{23} := \pi_2 \times \pi_3 \colon \Theta_k \to \mathbb{P}(\Lambda^k E_1) \times \mathbb{P}(\Lambda^{k+1} E_1)$ is the projection onto the two last factors, denote

$$\Gamma_k := \pi_{23}(\Theta_k) \subset \mathbb{P}(\Lambda^k E_1) \times \mathbb{P}(\Lambda^{k+1} E_1). \tag{2.2}$$

Given $0 \neq a \in E_1$, let $L_a^k \subset \Lambda^k E_1$ denote the image of the multiplication map $a \colon \Lambda^{k-1} E_1 \to \Lambda^k E_1$, and let $[L_a^k] \subset \mathbb{P}(\Lambda^k E_1)$ denote the corresponding projective linear subspace. If we write $\Lambda^k E_1$ as an internal direct sum $L_a^k \oplus V$ the complement $U_a = \mathbb{P}(\Lambda^k E_1) \setminus [L_a^k]$ is easily seen to be isomorphic to the total space of $\mathcal{O}_{\mathbb{P}(V)} \otimes L_a^k$, in other words, it is isomorphic to sum of $\mathcal{O}(1)$'s over the projective space $\mathbb{P}(V)$. It is easy to see that $\pi_1 \colon \mathcal{W}_k \to \mathbb{P}(E_1)$ is a fiber bundle whose fiber $\pi_1^{-1}([a])$ is isomorphic to U_a. In particular, we conclude that $\dim \Theta_k = \dim \mathcal{W}_k = \binom{n}{k} + n - 2$.

Now we consider the case $k = 1$. Here we can identify Γ_1 with the image of the flag variety $\mathbb{F}(1, 2; E_1)$ of lines in a plane in E_1 under the sequence of embeddings

$$\mathbb{F}(1, 2; E_1) \ \subset \ \mathbb{P}(E_1) \times G(2, E_1) \xrightarrow{Id \times \wp} \mathbb{P}(E_1) \times \mathbb{P}(\Lambda^2 E_1),$$

where $\wp$ is the Plücker embedding. Furthermore, the projection $\pi_{23}\colon \Theta_1 \to \mathbb{F}(1,2;E_1)$ is the evident $\mathbb{C}^*$-bundle.

The following observation is the key to computing the first resonance variety in terms of the Grassmann geometry described above. Let $I_k \subset \Lambda^k E_1$ denote the homogeneous component of degree k of the ideal I, and let $[I_k] \subset \mathbb{P}(\Lambda^k E_1)$ denote the corresponding linear subspace.

Proposition 2.1. *Using the notation in* (2.1)

$$R^1(\mathcal{A}) = \pi_1(\mu_1^{-1}(\ [I_2]\ ).$$

In other words, if $[I_2]^{dec} := G(2,E_1) \cap [I_2] \subset \mathbb{P}(\Lambda^2 E_1)$ *denotes the decomposable elements in* $[I_2]$*, then* $R^1(\mathcal{A}) = p_1(p_2^{-1}([I_2]^{dec})$*, where* p_1 *and* p_2 *denote the projections from* $\mathbb{F}(1,2;E_1)$ *onto* $\mathbb{P}(E_1)$ *and* $G(2,E_1)$*, respectively.*

Remark 2.2. In practice, a point of $\Upsilon := G(2,E_1) \cap [I_2]$ corresponds to a line in $\mathbb{P}(E_1)$. The resonance variety $R^1(\mathcal{A})$ is simply the collection $\cup_{L\in\Upsilon} G(1,L)$ of all lines in $\mathbb{P}(E_1)$ that correspond to points of Υ.

The situation with higher resonance varieties $R^k(\mathcal{A})$ is more complicated, but it still can be described as follows. Again, we refer to diagram (2.1) for notation.

Proposition 2.3. *The resonance variety* $R^k(\mathcal{A})$ *can be described as*

$$R^k(\mathcal{A}) = \overline{\pi_1\left(\ \mu_k^{-1}[I_{k+1}] \setminus \pi_2^{-1}[I_k]\right)},$$

where $\overline{\{\cdots\}}$ *denotes Zariski closure.*

3. Examples and Code

In this section, we compute several examples, comparing the time of the computation using the Grassmannian against the time of the computation using annihilator of Ext modules.

Example 3.1. We compute the first resonance variety of the A_3 arrangement of Example 1.2 using the approach of Proposition 2.1. First, we find $I_2^{\rm dec}$ observing that $u \in \Lambda^2 E_1$ is decomposable iff $u \wedge u = 0$, by the Grassmann-Plücker relations. Denote the basis of I_2 by $\rho_1 := \partial e_{145}$; $\rho_2 := \partial e_{012}$; $\rho_3 := \partial e_{034}$ and $\rho_4 := \partial e_{235}$ and, given $i \in \{0,\dots,5\}$ denote $\widehat{e}_i = e_0 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_5 \in \Lambda^5\mathbb{C}^6$.

A direct calculation gives

$$\begin{aligned} &\rho_1 \wedge \rho_2 = \partial\widehat{e}_3, \quad \rho_1 \wedge \rho_3 = -\partial\widehat{e}_2, \quad \rho_1 \wedge \rho_4 = \ \ \partial\widehat{e}_0, \\ &\rho_2 \wedge \rho_3 = \partial\widehat{e}_5, \quad \partial_2 \wedge \rho_4 = \ \ \partial\widehat{e}_4, \quad \rho_3 \wedge \rho_4 = -\partial\widehat{e}_1. \end{aligned}$$

Hence, given $0 \neq u = \sum_{i=1}^4 t_i\rho_i \in I_2$ one can write $0 = u \wedge u = \partial\omega$, where

$$\omega = t_1t_2\ \widehat{e}_3 - t_1t_3\ \widehat{e}_2 + t_1t_4\ \widehat{e}_0 + t_2t_3\ \widehat{e}_5 + t_2t_4\ \widehat{e}_4 - t_3t_4\ \widehat{e}_1.$$

Now, $\partial\omega = 0$ iff $\omega = \lambda\partial(e_{012345})$, for some λ, since $\Lambda^6\mathbb{C}^6$ is one-dimensional and $\partial\colon E \to E$ is acyclic. This gives

$$\lambda = t_1t_4 = t_3t_4 = -t_1t_3 = -t_1t_2 = t_2t_4 = -t_2t_3.$$

If $\lambda \neq 0$ then $t_i \neq 0$ for all i. In particular, since $t_1 \neq 0$ implies $t_2 = t_3 = -t_4$ and $t_2 \neq 0$ implies $t_3 = t_4 = -t_1$, one concludes that $u = t(\rho_1 + \rho_2 + \rho_3 - \rho_4)$ for some $t \neq 0$. It is easy to see that

$$\rho_1 + \rho_2 + \rho_3 - \rho_4 = (e_0 - e_1 - e_3 + e_5) \wedge (e_1 - e_2 + e_3 - e_4)$$

and that this decomposable vector corresponds precisely to the only essential component of $R^1(\mathcal{A})$.

If $\lambda = 0$ and $t_i \neq 0$, the equations above give $t_k = 0$ for all $k \neq i$. This gives the four additional elements ρ_i, $i = 1, 2, 3, 4$, in I_2^{dec} which correspond to the four local components.

```
i1 : load "Rscript"

i2 : time R1A A3
5*P
   0
     -- used .036 seconds
--The EPY script produces the module F(A) described in Section 1.

i3 : time ann(Ext^2(EPY(A3),S))
     -- used  0.125 seconds
```

The $ann(Ext^2(F(A), S))$ computation takes place in $\mathbb{P}(E_1)$, while the Grassmannian computation takes place in $\mathbb{P}(\Lambda^2(E_1))$. So the output $5 * P_0$ indicating that $R^1(A)$ consists of five points *means* five points in $G(2, E_1)$, so five lines in $\mathbb{P}(E_1)$.

The four-fold speedup seems small, but next we tackle a larger example.

Example 3.2. The Hessian arrangement consists of the twelve lines passing thru the nine inflection points of a smooth plane cubic curve. There are 4 lines incident at each of the nine inflection points, so that $R^1(A)$ will contain 9 local components, each of dimension two.

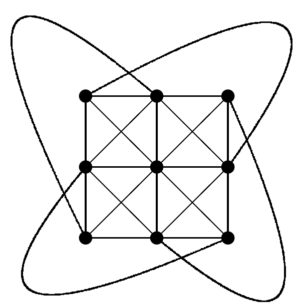

FIGURE 2. The Hessian arrangement

```
i4 : time R1A Hessian
54*P  + 10*P
    0       2
     -- used 14.004 seconds

--Again, we test against the time to find the annilihator of Ext^2.

o5 : time ann(Ext^2(EPY(hessian),S))
     -- used 9038.345 seconds
```

This computation indicates that for the Hessian configuration, $R^1(A)$ has ten components; in the Grassmannian, the tenth component is two-dimensional. Since it corresponds to a linear subvariety of $\mathbb{P}(E_1)$, and the space of lines in a fixed $\mathbb{P}^2$ is

two-dimensional, this means that (in contrast to the previous example), the non-local component of $R^1(A)$ is a $\mathbb{P}^2$. The Hessian configuration is the only example known with a nonlocal component of $R^1(A)$ of dimension greater than one, see [11].

The previous computations were performed on a ubuntu 7.10 system with a 2.2 GHz AMD processor and 64GB of RAM. We close with a short section illustrating how to implement this using the Macaulay2 package of Grayson and Stillman.

```
gring = (k,n)->(S = sort subsets(n,k);
                vlist = apply(S, i->w_i);
                ZZ/31991[vlist]);
--produce a ring where the variables are indexed by subsets, i.e. plucker ring.
--variables lex ordered in the indices.
---------------------------------------------------------------------------------

g2n=(n)->(G=gring(2,n);
          T=sort subsets(n,4);
          pluckers = ideal matrix {apply(T, i->
          w_{i#0,i#1}*w_{i#2,i#3}-w_{i#0,i#2}*w_{i#1,i#3}+w_{i#0,i#3}*w_{i#1,i#2})})
--script takes input n, and builds a ring with variables w_ij, return ideal
--of pluckers for affine G(2,n).
---------------------------------------------------------------------------------
OSrelns = (L)->(L1=apply(L, i->w_{i#0,i#1}-w_{i#0,i#2}+w_{i#1,i#2});
               L2 = jacobian matrix {L1})
--this takes a list of the rank 2 dependencies. For example, for A_3
--we have {{0,1,2},{0,3,4},{2,3,5},{1,4,5}}. Dependencies are decomposable
--two tensors, so give a point on the Grassmannian. The resulting matrix is
--the set of such points in P(\Wedge^2(K^n)).
---------------------------------------------------------------------------------
pointideal1 = (m)->(v=transpose vars G;
     minors(2,(v|m)))
--compute the ideal of a point.

pointsideal1 = (m)->(
     t=rank source m;
     J=pointideal1(submatrix(m, ,{0}));
     scan(t-1, i->( J=intersect(J,
     pointideal1(submatrix(m, ,{i+1})))));
     J)
--pointsideal1 takes a matrix with columns representing points, and returns
--the ideal of the points. So, to get the linear subspace spanned by the
--points, we'll need to take the degree one part of the ideal J.
---------------------------------------------------------------------------------
R1A = (M)->(t1=max mingle M;                              --determine n
            g2n(t1+1);                                    --build pluckers and ring
            P = pointsideal1(OSrelns M);                  --ideal of points on G(2,n)
            LL = select(P_*, f -> first degree f <= 1); --get the linear forms
            R1 = pluckers + ideal LL;                     --intersect G(2,n) with LL
            hilbertPolynomial coker gens R1)
--script to take the dependent sets of a matroid, then build G(2,n), find the
--linear span of the points of M on G(2,n), and intersect that linear span
--with G(2,n), yielding ideal of R^1(A) in G(2,n). Print Hilbert poly.
---------------------------------------------------------------------------------
```

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